# Многоборье-2014 (English)

## Regatta

### Junior League

#### First Round

**1.** Four natural numbers are given. For any triplet of numbers the following is known: their sum is divisible by the one that is middle-sized. Show that some of the numbers are equal.

*(A. Chapovalov)*

**2.** A circle with center $O$ is inscribed in the triangle $ABC$. The point $L$ lies on the extension of the side $AB$ beyond $A$. The tangent from $L$ intersects the side $AC$ in the point $K$. Find $\angle KOL$, if $\angle BAC = 50°$.

*(reformulation of a problem from Soros olympiad 1995)*

**3.** Around the table sit several knights, who always tell the truth, and knaves, who always lie. The first person said: "If you don't count me, there are one more knaves than knights here", the second one said: "If you don't count me, there are two more knaves than knights here", and so on until the last one. How many people could be sitting around the table?

*(A. Chapovalov)*

#### Second Round

**4.** All the numbers $a$, $b$, $c$ are different. All the lines $y = a^2x + bc$, $y = b^2x + ac$ and $y = c^2x + ab$ go through the same point. Prove that $a + b + c = 0$.

*(A. Chapovalov)*

**5.** From the point $A_0$ black and red rays are drawn with a $7°$ angle between them. Then a polyline $A_0A_1 \ldots A_{20}$ is drawn (possibly self-intersecting, but with all vertices different), in which all segments have length 1, all vertices with even numbers lie on the black ray and the ones with odd numbers — on the red ray. What is the index of the vertex that is farthest from $A_0$?

*(old Mathematical Kangaroo)*

**6.** Five Aborigines want to cross the river Limpopo in a boat which has place for two people. Initially, each of them has heard a rumour about somebody else, that this person is an Ebola virus carrier. About each person somebody has heard that rumour. If an Aborigine has heard the rumour about a person, he will not sit in the boat with him. The Aborigines do not talk on the banks of the river, but in the boat they exchange all rumours that are known to them. Is it possible that they (all five) will be able to cross the river anyway?

*(A. Chapovalov)*

## Third Round

**7.** There are 17 different cups on the table filled with compote. The total amount of dried fruit is 10% of all the compote. Petya and Vasya select and drink one glass each turn (Petya starts) until there is no compote left. Prove that Petya always can guarantee that his dried fruit share differs from 10% no more than the corresponding Vasya's share.

*(A. Chapovalov)*

**8.** In the convex quadrilateral $ABCD$, the bisectors of the angles $A$ and $C$ are parallel and intersect the diagonal $BD$ in two distinct points $P$ and $Q$, so that $BP = DQ$. Prove that the quadrilateral $ABCD$ is a parallelogram.

*(A. Chapovalov)*

**9.** Several rooks attack all the white squares of the chessboard $40 \times 40$. What is the largest number of black squares that can stay unattacked? (Rook attacks the square which it occupies.)

*(A. Chapovalov)*

## Fourth Round

**10.** 100 different integer numbers are written on a blackboard. Vasya replaces each number with either its square or its cube. Then Vasya does the process one more time, replacing each number with either its square or its cube (choosing the power independently each time). What is the smallest number of different numbers that can be written on the blackboard in the end?

*(A. Chapovalov inspired by additional problems from Mathematical Kangaroo 2014)*

**11.** A paper rectangle $ABCD$ ($AB = 3$, $BC = 9$) is folded in such way that vertices $A$ and $C$ coincide. What is the area of the obtained pentagon?

*(additional problems from Mathematical Kangaroo 2014)*

**12.** We call a nine-digit number *good* if it has a digit that can be moved to *another* place so that the new nine-digit number has its digits in a strictly ascending order. How many good numbers are there in total?

*(A. Chapovalov)*

# Senior League

## First Round

**13.** The numbers $a$ and $b$ are chosen in such a way that the graphs of $y = ax - b$ and $y = x^2 + ax + b$ enclose a finite figure of non-zero area. Prove that the origin lies inside this figure.

*(A. Chapovalov)*

**14.** All the vertices of a regular polygon lie on the surface of a cube, but its plane does not contain any of the cube's faces. What is the maximum possible number of vertices of such a polygon?

*(A. Chapovalov)*

**15.** Is it possible to cut a $12345 \times 6789$ squared rectangle along square borders into 7890 rectangles with equal diagonals?

*(A. Chapovalov)*

## Second Round

**16.** The product of all positive divisors of a natural number $n$ (including $n$) ends in 120 zeros. How many zeros can be at the end of the number $n$? (List all options and prove that no other exist.)

*(inspired by Mathematical Kangaroo 2013)*

**17.** The incircle of the triangle $ABC$ touches the side $AC$ at $D$. The $\angle BDC$ is equal to $60°$. Prove that the inscribed circles of triangle $ABD$ and $CBD$ touch $BD$ at the same point and find the ratio of the radii of these circles.

*(M. Volchkevich)*

**18.** 100 Aboriginal people were able to cross the river of Limpopo from the left to the right, in a boat which has place for two people. Initially, each Aborigine has heard a rumour about one or more of the others, that they are Ebola virus carriers. The Aborigine would not sit in the boat with somebody who he has heard this rumour about. On the left bank the spreading of the rumours is prohibited, but when the Aborigines reach the right bank, they come out of the boat, and everybody on that bank share all rumours with each other, and only then the boat returns. What is the least possible number of Aborigines that had no rumours that they are virus carriers about them at all?

*(A. Chapovalov)*

## Third Round

**19.** Baron Munchausen wrote 10 real numbers on the blackboard and wrote down their sum on a sheet of paper. With one operation he replaced one or more numbers on the board with their reciprocals, and again wrote down the sum on the same piece of paper. Is it possible that he wrote down the numbers $1, 2, \ldots, 500$ on the piece of paper as a result of 500 such operations?

*(A. Chapovalov)*

**20.** Let $ABC$ be an isosceles triangle with $AC = BC$. Let $N$ be a point inside the triangle such that $2\angle ANB = 180^\circ + \angle ACB$. Let $D$ be the intersection of the line $BN$ and the line parallel to $AN$ that passes through $C$. Let $P$ be the intersection of the angle bisectors of the angles $CAN$ and $ABN$. Show that the lines $DP$ and $AN$ are perpendicular.

*(Middle European 2013)*

**21.** For any coloring of the squares of the checkered board in black and white, the board is divided into connected coloured regions (in a chess colouring all regions consist of one square). Each step Petya selects one region and repaints it into the opposite colour. The repainted region is united together with the neighbouring regions of the same color, and the number of regions is reduced. What is the least number of steps that Petya needs to make the $13 \times 13$ board painted with chess pattern into a single-colour board?

*(A. Chapovalov)*

### Fourth Round

**22.** Find all the functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real $x$:

$$\underbrace{f(f(f \ldots f}_{13}(x) \ldots)) = -x, \qquad \underbrace{f(f(f \ldots f}_{8}(x) \ldots)) = x\,.$$

*(A. Ustinov)*

**23.** On the sides $BC$, $CA$ and $AB$ of an acute triangle $ABC$ the points $A_1$, $B_1$ and $C_1$ are chosen respectively. The circumscribed circles of the triangles $AB_1C_1$, $BC_1A_1$ and $CA_1B_1$ intersect at $P$. The points $O_1$, $O_2$ and $O_3$ are the centres of these circles. Prove that $4S(O_1O_2O_3) \geqslant S(ABC)$.

*(A. Smirnov inspired by Macedonia 2014)*

**24.** Prove that the number of ways to colour the edges of an $n$-gonal prism into 4 given colours, in such a way that the edges of each face have all of the colors, does not exceed $8 \cdot 6^{n-1} - 12 \cdot 2^{n-1}$.

*(A. Chapovalov)*

# Algebra and Number Theory

## Junior League

**25.** The train Moscow–Petushkí covers the initial distance — from Moscow to Drézna — in three times longer time than from Leónovo to Petushki. Also, the distance from Drezna to Petushki is covered twice as fast as from Moscow to Leonovo. How many times faster does the train cover the path from Moscow to Petushki compared to the path from Drezna to Leonovo?

*(South Africa 2014)*

**26.** For which natural numbers $n$ do there exist integers $x, y, z$ such that

$$x + y + z = 0, \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{n}\ ?$$

*(Iberoamerican 2011)*

**27.** Find all the perfect numbers which prime factorization satisfies the following property: every prime it contains is present in an odd power. A natural number is called perfect if it is equal to the sum all of its positive divisors smaller than the number itself — for example, $28 = 1 + 2 + 4 + 7 + 14$.

*(IMC 2014)*

**28.** For natural $n$, $k$ prove the inequality:

$$1 + \frac{1}{2} + \ldots + \frac{1}{nk} < \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) - \frac{1}{2}\,.$$

*(olympiad of the Faculty of Mathematics and Mechanics of SPBU)*

## Senior League

**29.** Solve the inequality

$$20 - 14(20 - 14(20 - \ldots - 14(20 - 14x)\ldots)) > x.$$

There are 2014 pairs of parentheses in total.

*(Soros olympiad 1995)*

**30.** Let $R(n)$ denote the number of ways to write a natural number $n$ as the sum $n = a + b$ of two primes (for example, $R(3) = 0$, $R(4) = 1$, $R(10) = 3$, since $10 = 3 + 7 = 5 + 5 = 7 + 3$). Show that if $p < q$ are two consecutive prime numbers, then the sum

$$2R(q-2) + 3R(q-3) + 5R(q-5) + \ldots + p \cdot R(q-p)$$

is divisible by $q$.

*(V. Bykovskii)*

**31.** Find all the functions $f(x)\colon \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > \frac{f(n)+f(f(n))}{2}$ for all natural $n$.

*(Iran 2014)*

**32.** Let $x_1 < x_2 < \ldots < x_n$ be an arithmetic progression of natural numbers, with $x_1$ and $x_2$ relatively prime. Let us assume that the product of the terms in the progression has only $m$ different prime divisors, where $m < n$. Prove that $x_1^{n-m} \leqslant (n-1)!$.

*(Paul Erdős)*

# Combinatorics and Logic

## Junior League

**33.** The surface of a $4 \times 4 \times 4$ cube is painted black and then the cube is sawed into 64 unit cubes. The faces of the cubes that have been sawed out are white. Is it possible to arrange the cubes into a parallelepiped $1 \times 8 \times 8$, with both $8 \times 8$ faces painted in a checkerboard pattern?

*(Högstadiets Matematiktävling 2002)*

**34.** There is a flashlight that has place for 2 batteries, and 10 batteries, 5 of which are good and 5 are bad. During one attempt, you can insert 2 batteries into the flashlight. It will only work if both inserted batteries are good. How is it possible to ensure that the flashlight will work on the 8th attempt at latest?

*(A. Ustinov)*

**35.** A regular 777-gon is given. Petya and Vasya take turns in drawing its diagonals, Petya starts. The drawn diagonals must not intersect inside the polygon. The person, whose newly drawn diagonal divides the polygon into parts where at least one part is a quadrilateral, loses. What player can win regardless of how his opponent plays?

*(Rioplatense Olympiad 2013, Level 3)*

**36.** In a company of $n$ people, there are no people that don't know anybody and no people that know everybody else. Prove that four of them can be chosen to seat at a round table so that everyone knows exactly one of his immediate neighbours.

*(Kürschák 2014)*

**37.** The magician and his assistant want to show the audience a magic trick. In the absence of the magician the audience writes down $n$ numbers on the sectors of a wheel, each of them equal to either 0 or 1. Then the assistant covers one of the numbers whith a black card after which the audience rotates the wheel. Then the magician comes in and he has to guess the number under the black card. For which $n$ can the magician and his assistant plan the magic trick so that its success is guaranteed?

*(A. Ustinov)*

## Senior League

**38.** A fly crawls along the edges of an octahedron, and, after creeping out of one vertex on an edge it always reaches the other vertex (and cannot change direction while moving along the edge). Is it possible that at one moment it has visited one vertex 2014 times and all the others — 650 times each?

*(Math League Tournaments)*

**39.** The magician and his assistant want to show the audience a magic trick. In the absence of the magician the audience writes down 9 numbers on the sectors of a wheel, each of them equal to either 0 or 1. Then the assistant covers one of the numbers whith a black card after which the audience rotates the wheel. Then the magician comes in and he has to guess the number under the black card. Can the magician and his assistant plan the magic trick so that its success is guaranteed?

*(A. Ustinov)*

**40.** At a school there are 200 underachieving and 200 overachievering students. On the New Years Eve Santa Claus brought a bag with 800 chocolate pralines. He want to give the pralines to the children so that the underachievers will get no more than one praline each, and the overachievers will get an even number of pralines each and at least 2. The headmaster wants to encourage the overachievers additionally by giving away 600 tangerines, at least one tangerine to each overachiever. Who has more ways to distribute the gifts and how many times more?

*(folklore)*

**41.** A class won a prize in form of a bag with 2014 coins, that the class chairman recieved (and all other students in the class have no money). Each time two classmates meet, they split the money they have equally among them if they have an even number of coins in total. If the sum is an odd number, they give one coin to their teacher, and the rest they split equally. After a while the teacher got all the coins. What is the minimum possible number of students in the class?

*(Skolornas Matematiktävling 2014)*

**42.** A diamond lies on each square of a chessboard, so that each pair of diamonds in the adjacent (along the side) squares differ in weight by less than 1 carat. Prove that these diamonds can be placed into the squares of the rectangle $2 \times 32$, so that diamonds in each pair of neighbouring squares still differ by less than 1 carat.

*(A. Chapovalov)*

# Geometry

## Junior League

**43.** In a right triangle $ABC$ with the right angle $B$ the bisector $CL$ is drawn. The point $L$ is equidistant from the points $B$ and the midpoint of the hypotenuse $AC$.

Find the angle $BAC$.

*(F. Nilov)*

**44.** In a convex quadrilateral $ABCD$ the equality $\angle BCA + \angle CAD = 180°$ holds. Prove that $AB + CD \geqslant AD + BC$.

*(A. Smirnov inspired by Serbian regional olympiad 2014)*

**45.** In a given trapezium $ABCD$ with $AB$ parallel to $CD$ and $AB > CD$, the line $BD$ bisects the angle $\angle ADC$. The line through $C$ parallel to $AD$ meets the segments $BD$ and $AB$ in $E$ and $F$, respectively. Let $O$ be the circumcenter of the triangle $BEF$. Suppose that $\angle ACO = 60°$. Prove the equality $CF = AF + FO$.

*(Middle European 2012)*

**46.** On the hypotenuse of a right triangle $ABC$ the points $M$ and $N$ are chosen in such a way that $AM < AN$. The line passing through $M$ that is perpendicular to $CN$ intersects the line $AC$ at the point $P$. The line passing through $N$ that is perpendicular to $CM$ intersects the line $BC$ at the point $Q$. Prove that the circumscribed circles of the triangles $APM$ and $BNQ$ and the line $PQ$ intersect at one point.

*(Iran 2014)*

## Senior League

**47.** A closed six-segmented polyline in space is given. Each of its segments is parallell to one of the orthogonal coordinate axes. Prove that its vertices lie on one sphere or in one plane.

*(Iran 2014)*

**48.** In a convex quadrilateral $ABCD$ the angles $\angle B$ and $\angle D$ are equal. It turned out that the intersection points of adjacent angle bisectors of $ABCD$ form a convex quadrilateral $EFGH$ ($E$ lies on the bisectors of the angles $\angle A$ and $\angle B$, $F$ — $\angle B$ and $\angle C$ etc.). Let $K$ be the intersection point of the diagonals of $EFGH$. Rays $AB$ and $DC$ intersect at the point $P$, rays $BC$ and $AD$ — at the point $Q$. Prove that $P$ lies on the circumcircle of the triangle $BKQ$.

*(Middle European 2012)*

**49.** The point $D$ is the midpoint of the bisector $BL$ in the triangle $ABC$. The points $E$, $F$ are chosen on the segments $AD$, $DC$ respectively, so that the angles $AFB$ and $BEC$ are right. Prove that the points $A$, $E$, $F$, $C$ lie on the same circle.

*(Ukraine 2014, problem 10.8)*

**50.** A set $M$ of points in the 3-dimensional space is called *interesting,* if for any plane there exist at least 100 points in $M$ outside this plane. For which minimal $d$ any interesting set contains an interesting subset with at most $d$ points?

*(IMC 2014)*

# Team Contest

## Junior League

**51.** A trapezoid $ABCD$ with bases $AB$ and $CD$ is such that the circumcircle of the triangle $BCD$ intersects the line $AD$ in a point $E$, distinct from $A$ and $D$. Prove that the circumcircle of the triangle $ABE$ is tangent to the line $BC$.

*(Baltic Way 2013)*

**52.** Is it possible to choose three numbers from $\frac{1}{100}, \frac{2}{99}, \frac{3}{98}, \ldots, \frac{100}{1}$ so that their product is equal to 1?

*(Högstadiets Matematiktävling 1997/1998)*

**53.** Numbers $1, 2, \ldots, n$ are written in some order in a single line. A pair of numbers is called a *hole* either if this numbers are next to each other or if all the numbers in between them are less than each of them. What is the maximum possible number of holes? (One number may be included in several holes.)

*(A. Lebedev, A. Chapovalov)*

**54.** Several chords are drawn in a circle so that every pair of them intersects inside the circle. Prove that all the draw cords can be intersected by the same diameter.

*(A. Chapovalov)*

**55.** Natural numbers $a$ and $b$ satisfy $2a^2 + a = 3b^2 - b$. Prove that $a + b$ is an exact square.

*(inspired by Skolornas Matematiktävling 2014 Qualification Round)*

**56.** The point $I_b$ is the center of an excircle of the triangle $ABC$, that is tangent to the side $AC$. Another excircle is tangent to the side $AB$ in the point $C_1$. Prove that the points $B$, $C$, $C_1$ and the midpoint of the segment $BI_b$ lie on the same circle.

*(inspired by olympiad of the Faculty of Mathematics and Mechanics of SPBU)*

**57.** Written on a blackboard is the polynomial $x^2 + x + 2014$. Sasha and Fedya take turns alternately (starting with Fedya) in the following game. At his turn, Fedya must either increase or decrease the coefficient of $x$ by 1. And at his turn, Sasha must either increase or decrease the constant coefficient by 1. Fedya wins if at any point in time the polynomial on the blackboard at that instant has integer roots. Prove that Fedya has a winning stratergy.

*(India 2014)*

**58.** On a circle 35 numbers are written. Every two adjacent numbers differ by no more than 1. Prove that the sum of squares of all the numbers is at least 10.

*(Mathlinks)*

*http://www.artofproblemsolving.com/Forum/viewtopic.php?t=612590*

**59.** For which $n$ can a convex $n$-gon be split into convex hexagons?

*(V. Bykovskii, A. Ustinov)*

## Senior League

**60.** Is it possible to choose five numbers from $\frac{1}{100}, \frac{2}{99}, \frac{3}{98}, \ldots, \frac{100}{1}$ so that their product is equal to 1?

*(Högstadiets Matematiktävling 1997/1998)*

**61.** Numbers $1, 2, \ldots, n$ are written in some order in a single line. A pair of numbers is called a *hole* either if this numbers are next to each other or if all the numbers in between them are less than each of them. What is the maximum possible number of holes? (One number may be included in several holes.)

*(A. Lebedev, A. Chapovalov)*

**62.** The point $I_b$ is the center of an excircle of the triangle $ABC$, that is tangent to the side $AC$. Another excircle is tangent to the side $AB$ in the point $C_1$. Prove that the points $B$, $C$, $C_1$ and the midpoint of the segment $BI_b$ lie on the same circle.

*(inspired by olympiad of the Faculty of Mathematics and Mechanics of SPBU)*

**63.** Prove that for any sequence of positive numbers $a_1, a_2, \ldots$ there exists a number $n$ such that $\frac{1+a_{n+1}}{a_n} \geqslant 1 + \frac{1}{n}$.

*(folklore)*

**64.** The spheres $S_1$, $S_2$ and $S_3$ are externally tangent to each other and all are tangent to some plane at the points $A$, $B$ and $C$. The sphere $S$ is externally tangent to the spheres $S_1$, $S_2$ and $S_3$ and is tangent to the same plane at the point $D$. Prove that the projections of the point $D$ onto the sides of the triangle $ABC$ are vertices of an equilateral triangle.

*(folklore)*

**65.** Through the center of an equilateral triangle $ABC$ an arbitrary line $l$ is drawn that intersects the sides $AB$ and $BC$ in points $D$ and $E$. A point $F$ is constructed, so that $AE = FE$ and $CD = FD$. Prove that the distance from $F$ to the line $l$ does not depend on the choice of $l$.

*(M. Volchkevich)*

**66.** For which $n$ can a convex $n$-gon be split into convex hexagons?

*(V. Bykovskii, A. Ustinov)*

**67.** The polynomial $f(x)$ has real coeffcients and is of degree $2n-1$, and $(f(x))^2 - f(x)$ is divisible by $(x^2 - x)^n$. Find all possible values of the leading coefficient of $f$.

*(North Countries Universities Mathematical Competition 2014)*

**68.** There are $n > 1$ balls in a box. Two players $A$ and $B$ are playing a game. At first, $A$ can take out $1 \leqslant k < n$ ball(s). When one player takes out $m$ ball(s), then the next player can take out $l$ balls, where $1 \leqslant l \leqslant 2m$. The person who takes out the last ball wins. Find all positive integers $n$ such that $B$ has a winning strategy.

*(Brazil 2014)*