1. The natural numbers from 1 to 2011 are written on a blackboard. Two players alternately erase one number each, until only two numbers, say p and q, remain. If none of the equations $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ has integer solutions, the first player wins. Can the second player prevent the first one from winning?

2. At least how many natural divisors can the number $p^2 + 2011$ have if p is prime?

3. Positive numbers a, b, c are such that the product of any two of them is greater than 1. Prove the inequality

$$\frac{a+b+1}{a^2+b^2+1} + \frac{b+c+1}{b^2+c^2+1} + \frac{a+c+1}{a^2+c^2+1} < \frac{2}{a+1} + \frac{2}{b+1} + \frac{2}{c+1}.$$

4. Prove that for each natural number m there is a natural number k for which $m \mid 3^{k+1} - 2^k - k$.

1. Let $p(x)$ be a polynomial with integer coefficients. Suppose that $p(1) = 2011$, $p(2011) = 1$ and $p(m) = m$ for some integer m. Find all possible values of m.

2. Define the *radical* $r(n)$ of a natural number n as the product of its distinct prime factors. For example, $r(2000) = 2 \cdot 5 = 10$ and $r(2011) = 2011$. A sequence (a_n) is given by its first term a_1 and the relation $a_{n+1} = a_n + r(a_n)$ for $n \geqslant 1$. Prove that there are a million consecutive terms in this sequence which form an arithmetic progression.

3. Positive numbers a, b, c satisfy $ab + bc + ca = 1$. Prove the inequality

$$\frac{(a+b)^2+1}{c^2+2} + \frac{(b+c)^2+1}{a^2+2} + \frac{(a+c)^2+1}{b^2+2} \geqslant 3.$$

4. Prove that there exist infinitely many pairs of natural numbers m, n such that $n! + 1$ is divisible by m, but $m - 1$ is not divisible by n.