IV International Zhautykov Olympiad in Mathematics
Almaty, 2008

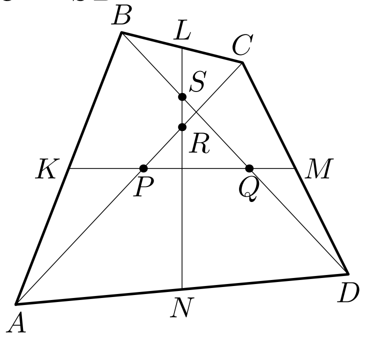
January 16, 2008. 9:00–13:00
First Day
(Each problem is worth 7 points.)

1. Points K, L, M, N are respectively the midpoints of sides AB, BC, CD, DA in a convex quadrilateral $ABCD$. Line KM meets diagonals AC and BD at points P and Q, respectively. Line LN meets diagonals AC and BD at points R and S, respectively.

Prove that if $AP \cdot PC = BQ \cdot QD$ then $AR \cdot RC = BS \cdot SD$.

Solution. First we prove that $\dfrac{AP}{PC} = \dfrac{BQ}{QD}$ in an arbitrary quadrilateral $ABCD$. Since the points K and M are the midpoints of AB and CD, we have $d(A, KM) = d(B, KM)$ and $d(C, KM) = d(D, KM)$ (here, $d(X, YZ)$ denotes the distance between point X and line YZ). Hence we have $\dfrac{AP}{PC} = \dfrac{d(A, KM)}{d(C, KM)} = \dfrac{d(B, KM)}{d(D, KM)} = \dfrac{BQ}{QD}$.

By analogous reasons, we get $\dfrac{AR}{RC} = \dfrac{DS}{SB}$.



Now, if $AP \cdot PC = BQ \cdot QD$, then $AP^2 = (AP \cdot PC) \cdot \dfrac{AP}{PC} = (BQ \cdot QD) \cdot \dfrac{BQ}{QD} = BQ^2$ and analogously $PC^2 = DQ^2$. Hence $AC = AP + PC = BQ + QD = BD$. Points R and S divide congruent diagonals AC and DB in the same ratio, hence $AR = DS$, $RC = SB$ and $AR \cdot RC = BS \cdot SD$.

2. A polynomial $P(x)$ with integer coefficients is called *good* if it can be represented as a sum of cubes of several polynomials (in variable x) with integer coefficients. For example, the polynomials $x^3 - 1$ and $9x^3 - 3x^2 + 3x + 7 = (x-1)^3 + (2x)^3 + 2^3$ are good.

a) Is the polynomial $P(x) = 3x + 3x^7$ good?

b) Is the polynomial $P(x) = 3x + 3x^7 + 3x^{2008}$ good?

Justify your answers.

Answer. a) Yes. b) No.

Solution. a) We present one of many possible examples. Note that $3x^7 + 3x^5 = (x^3 + x)^3 + (-x^3)^3 + (-x)^3$, $-3x^5 + 3x^4 = (x^2 - x)^3 + (-x^2)^3 + x^3$, $-3x^4 + 3x^2 = (x^2 - 1)^3 + (-x^2)^3 + 1^3$, $-3x^2 + 3x = (x-1)^3 + (-x)^3 + 1^3$. Hence, we get $3x + 3x^7 = (3x^7 + 3x^5) + (-3x^5 + 3x^4) + (-3x^4 + 3x^2) + (-3x^2 + 3x) = (x^3 + x)^3 + (x^2 - x)^3 + (x^2 - 1)^3 + (x-1)^3 + (-x^3)^3 + 2(-x^2)^3 + (-x)^3 + 2 \cdot 1^3$.

b) **Lemma.** Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be the polynomial with integer coefficients, and let $p(x)^3 = b_0 + b_1 x + \cdots + b_{3n} x^{3n}$. Then the number $(b_1 + b_2) + (b_4 + b_5) + \cdots + (b_{3n-2} + b_{3n-1})$ is even.

Proof. Consider the sum $B = (b_1 + b_2) + (b_4 + b_5) + \cdots + (b_{3n-2} + b_{3n-1})$. Note that

$$p(x)^3 = \sum_i a_i^3 x^{3i} + 3 \sum_{i \neq j} a_i^2 a_j x^{2i+j} + 6 \sum_{\substack{i \neq j \\ i \neq k \\ j \neq k}} a_i a_j a_k x^{i+j+k}.$$

The contribution of the first summand into B is 0, and the contribution of the third summand is even. So, it is sufficient to prove that the contribution of the second summand is even.

Note that $3 \mid i+2j$ if and only if $3 \mid 2i+j$, since both conditions are equivalent to $3 \mid i-j$. Hence, the contribution of the second summand into B is

$$\sum_{3 \nmid i-j} a_i^2 a_j = \sum_{\substack{i<j \\ 3 \nmid i-j}} (a_i^2 a_j + a_i a_j^2).$$

Each term in this sum is even, so the whole sum is even too, QED. $\square$

Suppose that the polynomial $c_0 + \cdots + c_m x^m$ is good. Then by Lemma the sum $(c_1+c_2)+(c_4+c_5)+\cdots+(c_{3n-2}+c_{3n-1})$ is even. Since the corresponding sum for the polynomial $3x+3x^7+3x^{2008}$ is odd, this polynomial cannot be represented in the desired form.

Comment. Suppose that the polynomial $b_0 + \cdots + b_n x^n$ is good. Then it is almost obvious that each coefficient of the form b_{3i+1} or b_{3i+2} is divisible by 3. Also, from Lemma, we have that $b_1+b_2+b_4+b_5+\ldots$ is even.

One can show that each polynomial with integer coefficients satisfying these two conditions is good.

3. Let $A = \{(a_1,\ldots,a_8) \mid a_i \in \mathbb{N},\ 1 \le a_i \le i+1 \text{ for each } i=1,\ldots,8\}$. A subset $X \subset A$ is called *sparse* if for each two distinct sequences $(a_1,\ldots,a_8),(b_1,\ldots,b_8) \in X$, there exist at least three indices i such that $a_i \ne b_i$.

Find the maximal possible number of elements in a sparse subset of set A.

Answer. $7! = 5040$.

Solution. First, we show that a sparse subset $X \subset A$ cannot contain more than 7! elements. For any $(a_1,\ldots,a_8) \in X$, consider its subsequence $(a_1,\ldots,a_6)$. If for two distinct elements $(a_1,\ldots,a_8),(b_1,\ldots,b_8) \in X$ their subsequences are identical, then they can differ only in 7th and 8th digits; it is impossible by the definition. Hence, for all elements of X, their subsequences are distinct; on the other hand, the number of possible subsequences is $2 \cdot 3 \cdot \cdots \cdot 7 = 7!$, hence $|X| \le 7!$.

Now, we will present an example of sparse subset consisting of 7! elements. Let X consist of all sequences of the form $(a_1,\ldots,a_8)$, where (i) $1 \le a_i \le i+1$ for each $i=1,\ldots,6$; (ii) $1 \le a_7, a_8 \le 7$; and (iii) $a_7 \equiv \sum_{i=1}^6 a_i \pmod 7$, $a_8 \equiv \sum_{i=1}^6 i a_i \pmod 7$. Obviously, there is exactly one such sequence for each $(a_1,\ldots,a_6)$, hence $|X| = 7!$. We will prove that this subset is sparse.

Consider two distinct sequences $(a_1,\ldots,a_8),(b_1,\ldots,b_8) \in X$. We claim that they differ in at least 3 digits. Note that $(a_1,\ldots,a_6) \ne (b_1,\ldots,b_6)$. If these subsequences differ in at least 3 digits, then the claim is trivial. If they differ in exactly one digit, then $\sum_{i=1}^6 a_i \not\equiv \sum_{i=1}^6 b_i \pmod 7$ and $\sum_{i=1}^6 i a_i \not\equiv \sum_{i=1}^6 i b_i \pmod 7$, hence $a_7 \ne b_7$, $a_8 \ne b_8$, and the sequences $(a_1,\ldots,a_8)$, $(b_1,\ldots,b_8)$ differ in 3 digits.

The only case left is when the subsequences $(a_1,\ldots,a_6)$ and $(b_1,\ldots,b_6)$ differ in exactly two digits, say $a_i \ne b_i$ and $a_j \ne b_j$. Then we need to show that $a_7 \ne b_7$ or $a_8 \ne b_8$. Assume the contrary. Then $a_i + a_j = b_i + b_j$ and $i a_i + j a_j = i b_i + j b_j$. It follows that $(i-j)a_i = (i a_i + j a_j) - j(a_i + a_j) = (i b_i + j b_j) - j(b_i + b_j) = (i-j) b_i$, and thus $a_i = b_i$ which is impossible. This contradiction finishes the proof.