IV International Zhautykov Olympiad in Mathematics
Almaty, 2008

January 17, 2008. 9:00–13:00
Second Day
(Each problem is worth 7 points.)

4. For each positive integer n, denote by $S(n)$ the sum of all digits in the decimal representation of n.

Find all positive integers n such that $n = 2S(n)^3 + 8$.

Answer. $n = 1, 2008, 13726$.

Solution. First, note that $n \equiv S(n) \pmod 9$. So, if $n = 2S(n)^3+8$, then $9 \mid 2S(n)^3 - S(n) - 1$, or $9 \mid (S(n)-1)(2S(n)^2+2S(n)+1)$. One can easily see that $2S(n)^2+2S(n)+1 = \frac{1}{2}((2S(n)+1)^2+1)$ is not divisible by 3, hence $9 \mid S(n) - 1$, and $S(n) = 9k + 1$ for some nonnegative integer k.

Let l be the number of digits in the decimal representation of n. Then $10^{l-1} \le n = 2S(n)^3+8 \le 2(9l)^3 + 8 \le 1500l^3$, so $10^{l-3} \le 15l^3$. On the other hand, one can easily show by the induction that $15l^3 < 10^{l-3}$ for $l \ge 7$. Thus $l \le 6$, and $S(n) \le 9l \le 54$. Hence $k \le 5$, and we are left to consider these cases.

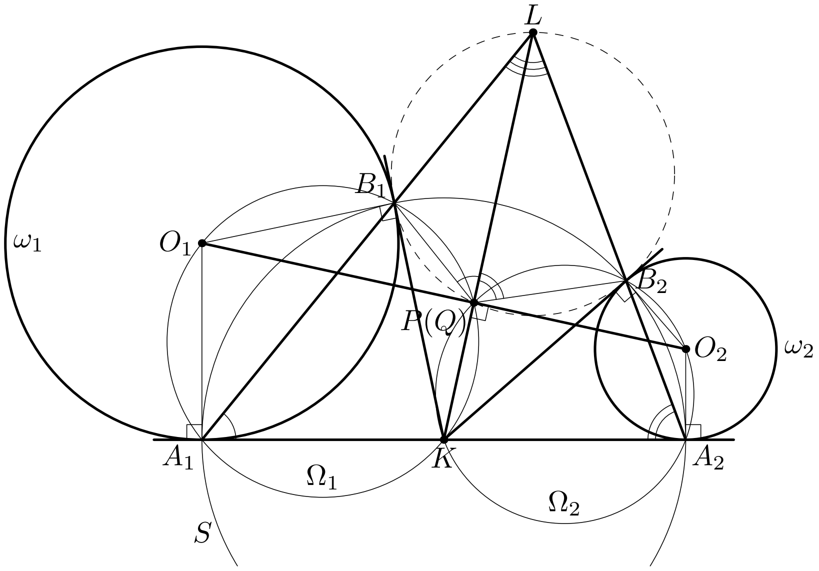
(i) If $k = 0$, then $n = 2S(n)^3 + 8 = 10$, which is a solution.
(ii) If $k = 1$, then $n = 2S(n)^3 + 8 = 2008$, which is a solution, too.
(iii) If $k = 2$, then $n = 2S(n)^3 + 8 = 13726$, which also is a solution.
(iv) If $k = 3$, then $n = 2S(n)^3 + 8 = 43912$, which is not a solution since $S(43912) \ne 28$.
(v) If $k = 4$, then $n = 2S(n)^3 + 8 = 101314$, which is not a solution since $S(101314) \ne 37$.
(vi) If $k = 5$, then $n = 2S(n)^3 + 8 = 194680$, which is not a solution since $S(194680) \ne 46$.

5. Nonintersecting circles ω_1 and ω_2 with centers O_1 and O_2 touch line ℓ at points A_1 and A_2, respectively (the circles lie on the same side of ℓ). Point K is the midpoint of segment A_1A_2. Points B_1 and B_2 are chosen on circles ω_1 and ω_2 respectively such that lines KB_1 and KB_2 touch ω_1 and ω_2 respectively (point B_1 is distinct from A_1, and point B_2 is distinct from A_2). Lines A_1B_1 and A_2B_2 meet in point L, and lines KL and O_1O_2 meet in point P.

Prove that points B_1, B_2, P, and L are concyclic.

Solution. Denote by Q the foot of perpendicular from K to line O_1O_2. Then points A_1, B_1, Q lie on the circle with diameter KO_1, while points A_2, B_2, Q lie on the circle with diameter KO_2. Denote these circles by Ω_1 and Ω_2, respectively. Moreover, points A_1, A_2, B_1, B_2 lie on the circle with center K; denote this circle by S.

Lines A_1B_1 and A_2B_2 are radical axes of circles Ω_1, S and Ω_2, S, respectively. Hence, point L is the radical center of circles Ω_1, Ω_2, S. So points L, Q and K lie on the radical axis of circles Ω_1 and Ω_2, and therefore Q is the point of intersection of KL and O_1O_2. Hence, we get $P = Q$.



Since quadrilaterals A_1B_1PK and A_2B_2PK are inscribed, we get $\angle LPB_1 = \angle LA_1K$ and $\angle LPB_2 = \angle LA_2K$. Hence, $\angle B_1PB_2 + \angle B_1LB_2 = (\angle LPB_1 + \angle LPB_2) + \angle A_1LA_2 = \angle LA_1A_2 + \angle LA_2A_2 + \angle A_1LA_2 = 180°$. Hence, points B_1, L, B_2, P are concyclic, QED.

Comment. It is easy to show that the circle B_1PB_2L is tangent to ω_1 and ω_2 at points B_1 and B_2.

6. Prove that for every positive real numbers a, b, c such that $abc = 1$, the inequality

$$\frac{1}{(a+b)b} + \frac{1}{(b+c)c} + \frac{1}{(c+a)a} \geq \frac{3}{2}$$

holds.

Solution.

Lemma. Suppose that $x_1 \geq x_2 \geq x_3$, $y_1 \leq y_2 \leq y_3$, and (y_1', y_2', y_3') is a permutation of (y_1, y_2, y_3). Then $x_1y_1' + x_2y_2' + x_3y_3' \geq x_1y_1 + x_2y_2 + x_3y_3$.

Proof. Since $y_1 \leq y_2 \leq y_3$, we get $y_1' \geq y_1$ and $y_1'+y_2' \geq y_1+y_2$; moreover, $y_1'+y_2'+y_3' = y_1+y_2+y_3$. Then we get

$$\begin{aligned} x_1y_1' + x_2y_2' + x_3y_3' &= (x_1 - x_2)y_1' + (x_2 - x_3)(y_1' + y_2') + x_3(y_1' + y_2' + y_3') \\ &\geq (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + x_3(y_1 + y_2 + y_3) = x_1y_1 + x_2y_2 + x_3y_3, \end{aligned}$$

QED. □

Denote by S the left-hand part of the desired inequality. Since S is invariant under the cyclical permutation of variables, we can assume that $a \leq b \leq c$ or $a \geq b \geq c$. In both cases, applying the Lemma we get

$$S = \frac{1}{(a+b)b} + \frac{1}{(b+c)c} + \frac{1}{(c+a)a} \geq \frac{1}{(a+b)c} + \frac{1}{(b+c)a} + \frac{1}{(c+a)b} = T.$$

Hence,

$$\begin{aligned} 2S \geq S + T &= \left(\frac{1}{(a+b)b} + \frac{1}{(a+b)c}\right) + \left(\frac{1}{(b+c)c} + \frac{1}{(b+c)a}\right) + \left(\frac{1}{(c+a)a} + \frac{1}{(c+a)b}\right) \\ &= \frac{b+c}{(a+b)bc} + \frac{c+a}{(b+c)ca} + \frac{a+b}{(c+a)ab} \geq 3\sqrt[3]{\frac{b+c}{(a+b)bc} \cdot \frac{c+a}{(b+c)ca} \cdot \frac{a+b}{(c+a)ab}} = 3, \end{aligned}$$

QED.